

FINITENESS PROPERTIES AND SUBGROUPS OF HYPERBOLIC GROUPS

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ABSTRACT. We present new constructions of subgroups of hyperbolic groups that satisfy certain finiteness properties. In particular, we present a new construction of a hyperbolic group with a finitely presented subgroup that is not of type F_3 .

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1. INTRODUCTION

Hyperbolic groups were introduced by Gromov [10] as a generalization of fundamental groups of negatively curved manifolds and of finitely generated free groups. A finitely generated group is hyperbolic if its Cayley graph is hyperbolic as a metric space. Hyperbolic groups are finitely presented, and have solvable word and conjugacy problems.

The property of being hyperbolic is not inherited by subgroups. For instance finitely generated free groups have infinitely generated subgroups which cannot be hyperbolic since hyperbolic groups are finitely presentable. It is natural to then ask if the property of being hyperbolic is inherited by finitely generated subgroups. Rips constructed the first examples of finitely generated subgroups of hyperbolic groups that are not finitely presentable.[11] So then one asks whether finitely presented subgroups inherit hyperbolicity.

Gromov gave what appeared to be an example of a non-hyperbolic finitely presented subgroup of a hyperbolic group. But it was later discovered by Bestvina that the ambient group of this example is not hyperbolic [3]. In 1999 Noel Brady constructed the first example of a hyperbolic group with a finitely presented subgroup that is not hyperbolic[3]—his subgroup is of type F_3 so cannot be hyperbolic since hyperbolic groups are of type F_∞ (see 2.2). The example was constructed as a fundamental group of a nonpositively curved cube complex whose universal cover admits a hyperbolic metric. The subgroup emerges as the kernel of a homomorphism to \mathbb{Z} . The cube complex X is a branched covering of a product of graphs, and the homomorphism to \mathbb{Z} is the induced homomorphism of a map $X \rightarrow S^1$ which lifts to a Morse function $\tilde{X} \rightarrow \mathbb{R}$. Bestvina–Brady Morse theory [2] is then used to prove the finiteness properties of the kernel. Brady asks if there are examples of hyperbolic groups with subgroups of type F_n but not type F_{n+1} for all $n \in \mathbb{N}$ [3].

We shall construct some new examples using Bestvina–Brady Morse theory. In Section 3 we present a novel example of a finitely generated but not finitely presentable subgroup of a $\text{CAT}(-1)$ group. Reading this construction will prepare the reader for the construction in Section 4. In Section 4, we shall construct a hyperbolic group with a finitely presented subgroup which is not of type F_3 . Our cube complexes will emerge as subcomplexes of products of graphs, and do not require branched coverings.

2. PRELIMINARIES

2.1. Cube Complexes and nonpositive curvature. By a *regular n -cube* \square^n we mean a cube in \mathbb{R}^n which is isometric to the cube $[0, 1]^n$ in \mathbb{R}^n . Informally a cube complex is a cell complex of regular Euclidean cubes glued along their faces by isometries. More formally, a cube complex is a cell complex X that satisfies the following conditions.

- (1) For each n -cell e in X there is an isometry $\chi_e : \square^n \rightarrow e$.
- (2) A map $f : \square^n \rightarrow e$ is an *admissible characteristic function* if it is χ_e precomposed with a partial isometry of \mathbb{R}^n . For any cell e in X the restriction of any χ_e to a face of \square^n is an admissible characteristic function of a cell of C .

The metric on such cube complexes is the piecewise Euclidean metric (see [5]).

Definition 1. *Given a face f of a regular cube \square^n , let x be the center of this face. The link $Lk(f, \square^n)$ is the set of unit tangent vectors at x that are orthogonal to f and point in \square^n . This is a subset of the unit sphere S^{n-1} which is homeomorphic to a simplex of dimension $n - \dim(f) - 1$. This admits a natural spherical metric, in which the dihedral angles are right angles.*

Definition 2. *Let f be a cell in X . Let $S = \{C \mid C \text{ is a cube in } X \text{ that contains } f \text{ as a face}\}$. The link $Lk(f, X) = \bigcup_{C \in S} Lk(f, C)$. This is a complex of spherical “all right” simplices glued along their faces by isometries. This admits a natural piecewise spherical metric.*

Gromov gave the following characterizations of $CAT(0)$ and $CAT(-1)$ cube complexes by combinatorial conditions on the links of vertices in [10]. A nice survey of these results can be found in [7] and [8] (see Proposition I.6.8).

Definition 3. *A simplicial complex is said to satisfy the no \square -condition if there are no 4-cycles in the 1-skeleton for which none of the pairs of opposite vertices of the cycle are connected by an edge. A simplicial complex Z is called a “flag” complex if any set v_1, \dots, v_n of vertices of Z that are pairwise connected by an edge span a simplex. This is also known as the “no empty triangles” condition.*

Definition 4. *A cube complex X is said to be nonpositively curved if the link of each vertex is a flag complex.*

Theorem 1. (Gromov) *A cube complex X is $CAT(0)$ if and only if it is nonpositively curved and simply connected. Furthermore, X admits a $CAT(-1)$ metric if and only if it is $CAT(0)$ and the link of each vertex satisfies the no \square -condition.*

Theorem 2. (Gromov, Eberlin, Bridson) *A $CAT(0)$ metric space with a cocompact group of isometries is hyperbolic if and only if it does not contain isometrically embedded flat planes.*

2.2. Topological finiteness properties of groups. The classical finiteness properties of groups are that of being finitely generated and finitely presented. These notions were generalized by C.T.C. Wall [12]. In this paper we are concerned with the properties *type F_n* . These properties are quasi-isometry invariants of groups [1]. In order to discuss these properties first we need to define Eilenberg-MacLane complexes.

Definition 5. *An Eilenberg-MacLane complex for a group G , or a $K(G, 1)$, is a connected CW-complex X such that $\pi_1(X) = G$ and \bar{X} is contractible.*

It is a fact that for any group G , there is an Eilenberg-MacLane complex X which is unique up to homotopy type. A group is said to be *of type F_n* if it has an Eilenberg MacLane

complex with a finite n -skeleton. Clearly, a group is finitely generated if and only if it is of type F_1 , and finitely presented if and only if it is of type F_2 . (For more details see [9].)

Torsion-free hyperbolic groups are of type F_∞ , which means that they are of type F_n for all $n \in \mathbb{N}$. This follows from the following result of Rips that appears in [11].

Theorem 3. (Rips) *Let H be a hyperbolic group. Then there exists a locally finite, simply connected, finite dimensional simplicial complex on which H acts faithfully, properly, simplicially and cocompactly. In particular, if H is torsion free, then the action is free and the quotient of this complex by H is a finite Eilenberg-MacLane complex $K(H, 1)$.*

2.3. Bestvina-Brady Morse theory. Here we shall sketch the main tool used in this paper. Bestvina-Brady Morse theory was introduced in [2] to study finiteness properties of subgroups of certain right angled Artin groups. Morse theory is defined more generally for affine cell complexes, but we shall only discuss the special case of piecewise euclidean cube complexes.

Let X be a piecewise Euclidean cube complex. Let G act freely, properly, cocompactly, cellularly and by isometries on X . Let \mathbb{Z} act on \mathbb{R} in the usual way. Let $\phi : G \rightarrow \mathbb{Z}$ be a homomorphism.

Definition 6. *A ϕ -equivariant Morse function is a map $f : X \rightarrow \mathbb{R}$ that satisfies,*

- (1) *For any cell F of X (with the characteristic function $\chi_F : \square^n \rightarrow F$), the composition $f \circ \chi_F$ is the restriction of a nonsingular affine map $\mathbb{R}^n \rightarrow \mathbb{R}$.*
- (2) *The image of the 0-skeleton is a discrete subset of \mathbb{R} .*
- (3) *f is G -equivariant, i.e for all $g \in G, x \in X, f(g \cdot x) = \phi(g) \cdot f(x)$.*

One can think of a Morse function as a height function on X . The kernel $H = \text{Ker}(\phi)$ acts on the level sets of X , i.e. inverse images $f^{-1}(x)$ for $x \in \mathbb{R}$. Topological properties of level sets can be used to deduce the finiteness properties of H . In [2] it was shown that the topological properties of the level sets are determined by the topology of links of vertices. We make this precise below.

Definition 7. *Given a vertex v of X , the ascending link $Lk^\uparrow(v, X)$ is defined as*

$$Lk^\uparrow(v, X) = \bigcup \{Lk(v, F) \mid v \text{ is the minima of } f \circ \chi_F\}$$

Similarly, the descending link Lk^\downarrow is define as

$$Lk^\downarrow(v, X) = \bigcup \{Lk(v, F) \mid v \text{ is the maxima of } f \circ \chi_F\}$$

We now summarize the main result of Bestvina-Brady Morse theory. For the details of the proof see [2] or [3].

Theorem 4. (Bestvina, Brady) *Let $n \geq 1$ be a fixed natural number. Let X be a CAT(0) cube complex, and G a group acting freely, properly, cocompactly and by isometries on X . Let \mathbb{Z} act in the usual way on \mathbb{R} , and $\phi : G \rightarrow \mathbb{Z}$ be a homomorphism. Let $f : X \rightarrow \mathbb{R}$ be a ϕ -equivariant Morse function and $H = \text{Ker}(\phi)$. Consider the following properties:*

- (1) *$Lk^\uparrow(v, X), Lk^\downarrow(v, X)$ are simply connected.*
- (2) *$\tilde{H}_k(Lk^\uparrow(v, X)), \tilde{H}_k(Lk^\downarrow(v, X)) = 0$ for $1 \leq k \leq n-1$ and $\tilde{H}_n(Lk^\uparrow(v, X), \mathbb{Z}), \tilde{H}_n(Lk^\downarrow(v, X), \mathbb{Z}) \neq 0$.*

If $n = 1$ and property (2) holds for all vertices in X , then H is finitely generated but not finitely presentable. If properties (1) and (2) hold for all vertices in X then H is of type F_n but not of type F_{n+1} .

3. A CAT(-1) EXAMPLE

In this section we produce a square complex X with a map $f : X \rightarrow S^1$ which lifts to an f_* -equivariant Morse function $\tilde{f} : \tilde{X} \rightarrow \mathbb{R}$ with the properties:

- (1) The link of each vertex is a finite graph with no 3 or 4 cycles,
- (2) The ascending and descending links of all vertices are homeomorphic to S^1 .

First we define a bipartite graph Γ which will be an ingredient in both constructions.

Definition 8. Γ is the following graph:

The vertex set: $V(\Gamma) = A_+ \cup A_- \cup B_- \cup B_+$ where,

- (1) $A_+ = \{a_0^+, a_1^+, \dots, a_{10}^+\}$.
- (2) $A_- = \{a_0^-, a_1^-, \dots, a_{10}^-\}$.
- (3) $B_+ = \{b_0^+, b_1^+, \dots, b_{10}^+\}$.
- (4) $B_- = \{b_0^-, b_1^-, \dots, b_{10}^-\}$.

The edge set: $E(\Gamma) = E_1 \cup E_2 \cup E_3$ where

- (1) E_1 consists of the edges $\{a_i^s, b_j^s\}$ for $s \in \{+, -\}$ and $i = j$ or $j = i + 1 \pmod{11}$.
- (2) E_2 consists of the edges $\{a_i^+, b_j^-\}$ for $j = i + 3 \pmod{11}$ or $j = i + 5 \pmod{11}$.
- (3) E_3 consists of the edges $\{a_i^-, b_j^+\}$ for $i = j$ or $j = i + 2 \pmod{11}$.

Lemma 5. Γ satisfies the following:

- (1) The subgraphs spanned by the vertex sets $A_+ \cup A_-$ and $B_+ \cup B_-$ have no edges, and $A_+ \cup B_+$, $A_+ \cup B_-$, $A_- \cup B_+$, $A_- \cup B_-$ span subgraphs that are each a cycle.
- (2) There are no 3-cycles or 4-cycles.

Proof. Γ is a bipartite graph so there are no 3-cycles. Property (1) in the statement of the lemma is easily verified. We claim that Γ has no 4-cycles. Assume there is a 4-cycle C . There are five cases to check. (We write a cycle as a set of edges.)

- (1) C lies in the subgraph spanned by $A_+ \cup A_- \cup B_-$.
So C is of the form $\{\{a_i^+, b_j^-\}, \{a_k^-, b_l^-\}, \{a_l^-, b_i^-\}, \{a_i^-, b_k^+\}\}$.
- (2) C lies in the subgraph spanned by $A_+ \cup A_- \cup B_+$.
So C is of the form $\{\{a_i^+, b_j^+\}, \{a_k^-, b_l^+\}, \{a_l^-, b_i^+\}, \{a_i^-, b_k^+\}\}$.
- (3) C lies in the subgraph spanned by $B_+ \cup B_- \cup A_-$.
So C is of the form $\{\{a_j^-, b_i^+\}, \{a_j^-, b_k^-\}, \{a_l^-, b_k^-\}, \{a_l^-, b_i^+\}\}$.
- (4) C lies in the subgraph spanned by $B_+ \cup B_- \cup A_+$.
So C is of the form $\{\{a_j^+, b_i^+\}, \{a_j^+, b_k^-\}, \{a_l^+, b_k^-\}, \{a_l^+, b_i^+\}\}$.
- (5) The vertices of C $\{v_1, \dots, v_4\}$ satisfy $v_1 \in A_+$, $v_2 \in B_+$, $v_3 \in A_-$, $v_4 \in B_-$.
So C is of the form $\{\{a_i^+, b_j^+\}, \{a_k^-, b_j^-\}, \{a_k^-, b_l^-\}, \{a_l^+, b_i^+\}\}$.

We treat cases (1) and (5). Cases (2), (3), (4) are similar to (1).

Case (1): Since the distinct vertices b_l^-, b_j^- share a neighbor a_k^- , we can assume without the loss of generality that $l = k$ and $j = k + 1 \pmod{11}$. So $|i - j| = \pm 1 \pmod{11}$. Now either $l = i + 3 \pmod{11}$ and $j = i + 5 \pmod{11}$, or $l = i + 5 \pmod{11}$ and $j = i + 3 \pmod{11}$. In either case we conclude that $|i - j| = \pm 2 \pmod{11}$. This is a contradiction. Therefore such a 4-cycle cannot exist.

Case (5): By construction we observe,

- (1) $j = i$ or $j = i + 1 \pmod{11}$.
- (2) $j = k$ or $j = k + 2 \pmod{11}$.
- (3) $l = k$ or $l = k + 1 \pmod{11}$.
- (4) $l = i + 3 \pmod{11}$ or $l = i + 5 \pmod{11}$.

From (1), (2), (3) above we deduce that if $|l - i| \equiv n \pmod{11}$, then $n \in \{-2, -1, 0, 1, 2\}$. From (4) on the other hand $|l - i| \equiv 3$ or $5 \pmod{11}$. This is a contradiction. Therefore we conclude that there are no 4-cycles in Γ . \square

Now we will define our square complex X . Let Θ_1 be a graph with vertices v_1, v_2 and the edge set $E(\Theta_1) = A_- \cup A_+ \subseteq V(\Gamma)$, such that each edge meets v_1 and v_2 . The A_- edges are oriented in the direction of v_2 (i.e. the arrow points to v_2) and the A_+ edges are oriented in the direction of v_1 . Similarly, let Θ_2 be a graph with vertices v_3, v_4 , and the edge set $E(\Theta_2) = B_- \cup B_+ \subseteq V(\Gamma)$. The B_- edges are oriented in the direction of v_4 and the B_+ edges are oriented in the direction of v_3 .

Consider the 2-complex $J = \Theta_1 \times \Theta_2$. The squares of J are ordered pairs (a, b) where $a \in A_+ \cup A_-$, $b \in B_+ \cup B_-$. We now define a subcomplex X of J in the following manner. Let $X^{(1)} = J^{(1)}$. Glue a 2-face along the boundary of every square in $X^{(1)}$ that has the property that the corresponding pair $\{a, b\}$ is an edge in Γ . The resulting square complex is X .

Identify S^1 with \mathbb{R}/\mathbb{Z} . The orientations on the edges of Θ_1, Θ_2 determine maps $l_i : \Theta_i \rightarrow S^1$. Under this map each vertex maps to $[0]$, and the map on each edge is as follows: Identify the edge with the unit interval $[0, 1]$ in a way that the edge is oriented towards the vertex identified with 1. Then define the map l_i on an edge as $x \mapsto [x]$. Now identify each square C of X isometrically with the unit square $[0, 1]^2$, where the edges $\{v\} \times [0, 1]$ and $[0, 1] \times \{v\}$ are oriented towards $(v, 1)$ and $(1, v)$ respectively for each $v \in \{0, 1\}$. The map f on C is now defined as $(x, y) \mapsto [x + y]$.

The map $f_* : \pi_1(X) \rightarrow \pi_1(S^1)$ is the induced map on the fundamental groups, and $\pi_1(X)$ acts on the universal cover \tilde{X} by deck transformations. f lifts to a map $\tilde{f} : \tilde{H} \rightarrow \mathbb{Z}$, which is a f_* -equivariant Morse function on \tilde{X} . Conditions (1), (2), (3) of the definition of a Morse function are apparent, and condition (4) follows from the definition of the lift \tilde{f} .

Theorem 6. *The square complex X has the following properties.*

- (1) \tilde{X} admits a CAT(-1) metric.
- (2) $\text{Ker}(f_* : \pi_1(X) \rightarrow \pi_1(S^1))$ is finitely generated but not finitely presented.

Proof. By construction, the link of each vertex is homeomorphic to the graph Γ . Since Γ is a flag simplicial complex with no empty-squares, (1) follows from Theorem 1. By construction the ascending and descending links of each vertex are homeomorphic to S^1 . Therefore (2) follows from Theorem 4. \square

4. A SUBGROUP OF TYPE F_2 BUT NOT TYPE F_3

We shall construct a three-dimensional cube complex Δ and a Piecewise Linear function $g : \Delta \rightarrow S^1$ that lifts to a g_* equivariant Morse function $\tilde{g} : \tilde{\Delta} \rightarrow \mathbb{R}$ satisfying the hypothesis of Theorem 4 with $n = 2$. It will be shown that $\tilde{\Delta}$ is a hyperbolic metric space. The cube complex Δ will be constructed as a subcomplex of a product of finite graphs.

We first define graphs U, V, W , each of which is isomorphic to $K_{22,22}$, the complete bipartite graph with 22 vertices in each “part”. Let the parts of U, V, W be U_1, U_2, V_1, V_2 and W_1, W_2 respectively. The vertices of U_1, V_1, W_1 are $\{a_0^+, \dots, a_{10}^+, a_0^-, \dots, a_{10}^-\}$ and the vertices of U_2, V_2, W_2 are $\{b_0^+, \dots, b_{10}^+, b_0^-, \dots, b_{10}^-\}$.

For each of the graphs U, V, W , we fix the following orientations on edges. Given an edge $\{a_n^s, b_m^t\}$ for $0 \leq m, n \leq 10, s, t \in \{+, -\}$, if $s = t$ then the edge is oriented toward a_n^s otherwise the edge is oriented toward b_m^t . So any vertex of U, V, W has 11 incoming and 11 outgoing edges. Declare each edge of U, V, W to be isometrically identified with the unit interval $[0, 1]$ in such a way that the edge is oriented towards the vertex identified with 1. Let $U \times V \times W$ be the product cube complex. We define a cube subcomplex Δ as follows.

Definition 9. Let (u, v, w) be a vertex in $U \times V \times W$. Then $(u, v, w) \in \Delta$ if one of the following holds. (Recall that Γ is the graph defined in section 3.)

- (1) For some $i \in \{1, 2\}$, $u \in U_i, v \in V_i, w \in W_i$.
- (2) $u \in U_1, v \in V_2$ and $\{u, v\}$ is an edge in Γ .
- (3) $v \in V_1, w \in W_2$ and $\{v, w\}$ is an edge in Γ .
- (4) $u \in U_2, w \in W_1$ and $\{u, w\}$ is an edge in Γ .

We declare a cell of $U \times V \times W$ to be in Δ if all its incident vertices are in Δ .

It follows immediately that Δ is a piecewise Euclidean cube complex. Vertices in $U \times V \times W$ that satisfy (1) above are said to be *type 1* vertices, and vertices that satisfy either (2), (3) or (4) are said to be *type 2* vertices. It is an easy exercise to show that any two vertices in Δ are connected by a path in $\Delta^{(1)}$. (Check this for type 1 vertices first.) We conclude that Δ is connected.

The graph Ω of Figure 1 serves as a tool for determining when a given vertex is in Δ . The edges of the graph encode the conditions of the definition above as follows: Given a vertex $\tau = (u, v, w)$ such that $u \in U_i, v \in V_j, w \in W_k$, either τ is a type 1 vertex (and hence is in Δ), or there is an edge connecting two of the three nodes U_i, V_j, W_k in the graph Ω . Then $\tau \in \Delta$ if and only if the corresponding pair from u, v, w forms an edge in Γ .

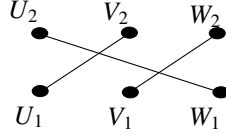


FIGURE 1. The graph Ω

One observes that the map $U \times V \times W \rightarrow U \times V \times W; (u, v, w) \mapsto (w, u, v)$ and its iterates are cell permuting isometries whose restriction to Δ induces a map $\Delta \rightarrow \Delta$ that is a cell permuting isometry. These symmetries of our construction will be invoked in arguments that follow.

The orientations on the edges induce a PL function $f : \Delta \rightarrow S^1$, which is $x \rightarrow [x]$ on each edge, and on the product it is defined as $(x, y, z) \rightarrow [x+y+z]$. (Recall the identification of each edge with $[0, 1]$ that was described above, and the identification of S^1 with \mathbb{R}/\mathbb{Z} .) The map f lifts as a PL Morse function between the universal covers $\tilde{f} : \tilde{\Delta} \rightarrow \mathbb{R}$. Now we prove two key lemmas, the first of which examines the links of vertices.

Lemma 7. Let $\tau \in \Delta$ be a vertex. Then the following holds. (Here \star denotes the topological join and X is a discrete set of four points.)

- (1) If τ is a type 1 vertex then $Lk(\tau, \Delta)$ is homeomorphic to $X \star X \star X$.
- (2) If τ is a type 2 vertex then $Lk(\tau, \Delta)$ is homeomorphic to $\Gamma \star X$.

In particular, Δ is nonpositively curved. Furthermore, in either case $Lk^\dagger(\tau, \tilde{\Delta}), Lk^\perp(\tau, \tilde{\Delta})$ are both homeomorphic to S^2 .

Proof. Let $\tau = (u, v, w)$ be a type 1 vertex. Assume that $u \in U_1, v \in V_1, w \in W_1$. Recall that $u, v, w \in \{a_0^+, \dots, a_{10}^+, a_0^-, \dots, a_{10}^-\} \subseteq V(\Gamma)$. Let the neighbors of u, v, w in Γ be

$$\{b_{k_1}^+, b_{k_2}^+, b_{k_3}^-, b_{k_4}^-\}, \{b_{l_1}^+, b_{l_2}^+, b_{l_3}^-, b_{l_4}^-\}, \{b_{j_1}^+, b_{j_2}^+, b_{j_3}^-, b_{j_4}^-\}$$

respectively. The following are the 1-cells adjacent to τ in Δ :

- (1) $[u, b_{j_i}^+] \times v \times w$ for $1 \leq i \leq 2$ and $[u, b_{j_i}^-] \times v \times w$ for $3 \leq i \leq 4$.

- (2) $u \times [v, b_{k_i}^+] \times w$ for $1 \leq i \leq 2$ and $u \times [v, b_{k_i}^-] \times w$ for $3 \leq i \leq 4$.
- (3) $u \times v \times [w, b_{l_i}^+]$ for $1 \leq i \leq 2$ and $u \times v \times [w, b_{l_i}^-]$ for $3 \leq i \leq 4$.

It follows that $Lk(\tau, \Delta \cap (U \times v \times w))$, $Lk(\tau, \Delta \cap (u \times V \times w))$, $Lk(\tau, \Delta \cap (u \times v \times W))$ are all discrete sets of four points each. Furthermore, it follows from the definition of Δ that $[u, p] \times [v, q] \times [w, r]$ is a cube in Δ for each $p \in \{b_{j_1}^+, b_{j_2}^+, b_{j_3}^-, b_{j_4}^-\}$, $q \in \{b_{k_1}^+, b_{k_2}^+, b_{k_3}^-, b_{k_4}^-\}$, $r \in \{b_{l_1}^+, b_{l_2}^+, b_{l_3}^-, b_{l_4}^-\}$. So $Lk(\tau, \Delta)$ is the topological join of these sets. Observe that exactly two of the four 1-cells in each of (1), (2), (3) are oriented away from τ and the remaining are oriented towards τ . So it follows immediately that $Lk^\uparrow(\tau, \Delta)$, $Lk^\downarrow(\tau, \Delta)$ are homeomorphic to S^2 . The case where $u \in U_2, v \in V_2, w \in W_2$ is similar.

Now consider the case where $\tau = (u, v, w) \in \Delta$ is a type 2 vertex. Assume that $u \in U_2, v \in V_1, w \in W_1$. The 1-cells incident to τ in Δ are:

- (1) $[u, a_i^s] \times v \times w$ for $0 \leq i \leq 10, s \in \{+, -\}$.
- (2) $u \times [v, b_i^s] \times w$ for $0 \leq i \leq 10, s \in \{+, -\}$.
- (3) $u \times v \times [w, p]$, for $p \in \{b_{n_1}^+, b_{n_2}^+, b_{n_3}^-, b_{n_4}^-\}$ where $\{b_{n_1}^+, b_{n_2}^+, b_{n_3}^-, b_{n_4}^-\}$ are the four neighbors of v in Γ .

Now given 1-cells $[u, a_i^s] \times v \times w, u \times [v, b_j^t] \times w$, observe that there is a square $[u, a_i^s] \times [v, b_j^t] \times w$ in Δ if and only if a_i^s, b_j^t are connected in Γ by an edge. This means that $Lk(\tau, \Delta \cap (U \times V \times w)) \cong \Gamma$. Furthermore, the definition of Δ implies that $[u, a_i^s] \times [v, b_j^t] \times [w, b_k^r]$ is a cube in Δ if and only if $[u, a_i^s] \times [v, b_j^t] \times w$ is a square in Δ and $u \times v \times [w, b_k^r]$ is a 1-cell in Δ . This means that $Lk(\tau, \Delta)$ is the topological join of Γ and the discrete set of four points, $Lk(\tau, \Delta \cap (u \times v \times W))$.

Now $Lk^\uparrow(\tau, \Delta \cap (U \times V \times w))$, $Lk^\downarrow(\tau, \Delta \cap (U \times V \times w))$ are both cycles, and $Lk^\uparrow(\tau, \Delta \cap (u \times v \times W))$, $Lk^\downarrow(\tau, \Delta \cap (u \times v \times W))$ are both discrete sets of two points each. This means that $Lk^\uparrow(\tau, \Delta) \cong S^2$ and $Lk^\downarrow(\tau, \Delta) \cong S^2$. For any other vertex τ of type 2, the analysis is similar by symmetry of the construction. \square

So far we have shown the following.

- (1) Δ is a nonpositively curved cube complex.
- (2) By Lemma 7 and Theorem 4 it follows that $\text{Ker}(f_* : \pi_1(\Delta) \rightarrow \pi_1(S^1))$ is finitely presented but not of type F_3 .

Now we will show that $\tilde{\Delta}$ is a hyperbolic metric space and hence $\pi_1(\Delta)$ is a hyperbolic group. We have already established that $\tilde{\Delta}$ is a $CAT(0)$ space, and so by Theorem 2 it suffices to show that $\tilde{\Delta}$ does not contain isometrically embedded flat planes.

Let $\tau = (u, v, w)$ be a type 2 vertex in Δ . From the previous lemma $Lk(\tau, \Delta) \cong \Gamma \star X$, where X is a discrete set of four points. So $Lk(\tau, \Delta^{(1)})$ is naturally identified with $V(\Gamma) \cup X$.

Lemma 8. *Let τ, τ' be type 2 vertices in Δ such that $[\tau, \tau']$ is a 1-cell in Δ . The $Lk(\tau, [\tau, \tau']) \in X$ if and only if $Lk(\tau', [\tau, \tau']) \in X$.*

Proof. We assume that $\tau = (u, v, w), \tau' = (u', v, w)$. Let $u \in U_i, u' \in U_j$ where $\{i, j\} = \{1, 2\}$. Also, let $v \in V_k, w \in W_l$. Assume that $Lk(\tau, [\tau, \tau']) \in X$. It follows that $Lk(\tau, \Delta \cap (U \times v \times w)) = X$. So U_j is connected by an edge in Ω with either V_k or W_l . We will show that U_i is connected by an edge with either V_k or W_l in Ω , and hence $Lk(\tau', [\tau, \tau']) \in X$.

Assume that this is not the case. Then since τ' is a type 2 vertex it must be the case that V_k, W_l are connected by an edge in Ω . This cannot be true since U_j is connected by an edge with either V_k or W_l . This proves our assertion. By symmetry of our construction this follows for any arbitrary type 2 vertex in Δ . \square

Definition 10. A 1-cell $[\tau, \tau']$ in Δ satisfying the statement of Lemma 8 i.e., $Lk(\tau, [\tau, \tau']) \in X$ and $Lk(\tau', [\tau, \tau']) \in X$ is called a special 1-cell. Denote the union of all special 1-cells in Δ by L . A lift of a special 1-cell in $\tilde{\Delta}$ is a special 1-cell in $\tilde{\Delta}$ and \tilde{L} is the union of all special 1-cells in $\tilde{\Delta}$.

Figure 2 depicts a cube in Δ . The three bold 1-cells are the special 1-cells, the vertices τ_1, τ_2 are the type 1 vertices and the remaining vertices are of type 2.

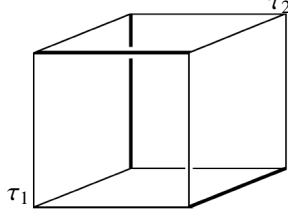


FIGURE 2.

Lemma 9. $\tilde{\Delta}$ does not contain isometrically embedded flat planes, and hence is a hyperbolic metric space.

Proof. The proof is similar to the proof of 6.1(3) in [3], and we claim no originality here. We adapt that argument to our construction. Let us assume that there is an isometric embedding $i : \mathbb{R}^2 \rightarrow \tilde{\Delta}$. We say that a point $x \in i(\mathbb{R}^2)$ is a *transverse intersection point* if there is a neighborhood U of x in $i(\mathbb{R}^2)$ such that the intersection of U with \tilde{L} is the point x . Following [3], our proof is divided into two steps. In *step 1* we will show that $i(\mathbb{R}^2)$ has a transverse intersection point. In *step 2* we will show that the angle around the transverse intersection point in $i(\mathbb{R}^2)$ is greater than 2π , contradicting the fact that this is an isometric embedding.

Step 1: Let C be a cube in $\tilde{\Delta}$ such that $i(\mathbb{R}^2) \cap C$ is nonempty and two dimensional. (Such cubes must exist in $\tilde{\Delta}$ since $i(\mathbb{R}^2)$ is an isometric embedding.) There are four cases to consider.

In the first case, $i(\mathbb{R}^2)$ intersects a special 1-cell e of C in a vertex p . Now $i(\mathbb{R}^2)$ must also intersect a neighboring cube C' of C that shares a 2-face with C and contains a special 1-cell e' incident to p . Then either $i(\mathbb{R}^2)$ contains e' or p is a transverse intersection point. Since $i(\mathbb{R}^2)$ is an isometric embedding, it cannot contain e' or else it would also contain e . In the second case, $i(\mathbb{R}^2)$ intersects a special 1-cell of C in an interior point, in which case it is clear that this is a transverse intersection point. In the third case, $i(\mathbb{R}^2)$ contains a special 1-cell of C . In this case it transversely intersects a different special 1-cell of C . (Recall that $i(\mathbb{R}^2) \cap C$ is two dimensional and see Figure 2).

Finally, consider the case where $i(\mathbb{R}^2)$ does not intersect a special 1-cell of C . Then $i(\mathbb{R}^2)$ intersects a 1-cell incident to a type 1 vertex τ in C . Let J be the subcomplex of $\tilde{\Delta}$ consisting of cubes in $\tilde{\Delta}$ that have a nonempty intersection with $i(\mathbb{R}^2)$. Recall that $Lk(\tau, \tilde{\Delta}) \cong X \star X \star X$ where X is a discrete set of four points. The set of cubes incident to τ in J is a subcomplex of a stack of 8 cubes depicted in Figure 3. (C is one of the 8 cubes.) Here the bold 1-cells are the special 1-cells. Now by figure 3 it must be the case that $i(\mathbb{R}^2)$ transversely intersects a special 1-cell in a neighboring cube of C .

Step 2: Now we demonstrate a contradiction to our assumption that $i(\mathbb{R}^2)$ is an isometrically embedded flat plane in $\tilde{\Delta}$. This involves computing the angle in $i(\mathbb{R}^2)$ around a transverse intersection point p .

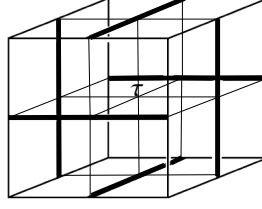


FIGURE 3.

Let Y be the subcomplex consisting of cubes in $\tilde{\Delta}$ that contain p and for which $C \cap i(\mathbb{R}^2)$ is 2-dimensional. For each cube C in Y , $i^{-1}(i(\mathbb{R}^2) \cap C)$ is a polygon, and $i^{-1}(i(\mathbb{R}^2) \cap Y)$ is a union of polygons in \mathbb{R}^2 that are incident to $i^{-1}(p)$ such that the sum of the angles at $i^{-1}(p)$ in each polygon is 2π .

Let C, C' be cubes in Y such that $i^{-1}(i(\mathbb{R}^2) \cap C), i^{-1}(i(\mathbb{R}^2) \cap C')$ are adjacent polygons. Then $i(\mathbb{R}^2) \cap (C \cup C')$ is locally the intersection of $i(\mathbb{R}^2)$ with a Euclidean half space. This means that the angle sum of any two consecutive polygons in $i^{-1}(i(\mathbb{R}^2) \cap Y)$ is π . To establish a contradiction, it suffices to show that the number of polygons in $i^{-1}(i(\mathbb{R}^2) \cap Y)$ is greater than 4. Let $e = [\tau, \tau']$ be a special 1-cell containing p . Recall that since τ is a type 2 vertex, $Lk(\tau, \tilde{\Delta})$ is naturally identified with $\Gamma \star X$. Now $Lk(\tau, Y)$ is a subcomplex of $\Gamma \star X$ and by Lemma 8 we know that $Lk(\tau, [\tau, \tau']) \in X$. So there is a natural bijection between the aforementioned set of polygons and the set of edges of a cycle in Γ . Since all cycles in Γ have more than four edges, we have established that the angle around p in $i(\mathbb{R}^2)$ is greater than 2π contradicting the fact that this is an isometric embedding. \square

5. CONCLUDING REMARKS

At this point we do not have a concrete way of distinguishing our example in Section 4 from Brady's example in [3], other than the method of construction. In fact, there is a striking similarity between the two examples, even though the methods of construction are entirely different. Nevertheless, we do believe that our approach is less abstract.

This construction does not seem to have a natural generalization in higher dimensions. It seems likely that any natural generalization in dimensions four and higher always produces flat planes in the universal cover. As a result, it is not clear whether such an approach can be used to construct hyperbolic groups with subgroups that are of type F_n but not of type F_{n+1} for $n > 2$.

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